ON CONNECTEDNESS IN TWO-WAY ELIMINATION OF HETEROGENEITY DESIGNS

By D. Raghavarao* and W. T. Federer

Cornell University

BU-467-M July, 1973

ABSTRACT

A necessary and sufficient condition is established for doubly-connectedness in b-row and k-column designs in which all cells are filled. An algorithm is presented for constructing a class of doubly-disconnected designs which are pairwise connected with respect to rows, columns, and treatments. A necessary condition for doubly-connectedness in a generalized two-way elimination of heterogeneity designs is provided and a property of doubly-connected designs is given.

*On leave from Punjab Agricultural University, India.

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1. Introduction. Let the experimental material be arranged in b rows and k columns and let v treatments be applied to the experimental units, the i^{th} treatment being replicated r_i times $(i = 1,2,\cdots,v)$. Let $u_1,u_2,\cdots,u_b$ units be treated with the treatments in the b rows and let $w_1,w_2,\cdots,w_k$ units be treated with the treatments in the k columns. If $u_1 = u_2 = \cdots = u_b = k$ and $w_1 = w_2 = \cdots w_k = b$ we obtain the usual two-way elimination of heterogeneity designs, otherwise we obtain generalized two-way elimination of heterogeneity designs.

Let $N = (n_{ij})$ be a $b \times k$ matrix, where n_{ij} is the number of treated units in the i^{th} row and j^{th} column, $L = (\ell_{ij})$ be a $v \times b$ matrix, where ℓ_{ij} is the number of treated units with the i^{th} treatment in the j^{th} row and $M = (m_{ij})$ be a $v \times k$ matrix, where m_{ij} is the number of treated units with the i^{th} treatment in the j^{th} column. Let A^- denote a generalized inverse of A. Let C_1,C_2,C_3,C_4, C^*,C_3^*,C_4^* be the well-known C matrices for the following purposes:

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*On leave from Punjab Agricultural University, India.

†This work was supported by NIH Research Grant GM-05900.

Key words: doubly-connected design,
two-way design,
pairwise connected design.

Matrix	Estimating	Eliminating	Ignoring
$C_1 = \mathrm{diag}(r_1, r_2, \cdots, r_v) - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) L'$	treatment effects	row effects	column effects
$C_2 = \mathrm{diag}(r_1, r_2, \cdots, r_v) - M\,\mathrm{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) M'$	treatment effects	column effects	row effects
$C_3 = \mathrm{diag}(w_1, w_2, \cdots, w_k) - N'\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N$	column effects	row effects	treatment effects
$C_4 = \mathrm{diag}(u_1, u_2, \cdots, u_b) - N\,\mathrm{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) N'$	row effects	column effects	treatment effects
$C^* = C_1 - \left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right) \times C_3^{-}\left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)$	treatment effects	row and column effects	—
$C_3^* = C_3 - \left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)' \times C_1^{-}\left(M - L\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)$	column effects	row and treatment effects	—
$C_4^* = C_4 - \left(L - M\,\mathrm{diag}\left(\frac{1}{w_1}, \frac{1}{w_2}, \cdots, \frac{1}{w_k}\right) N'\right)' \times C_1^{-}\left(L - M\,\mathrm{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N'\right)$	row effects	column and treatment effects	—

C^* in case of an ordinary two-way elimination of heterogeneity designs reduces to the following:

$$C^* = \mathrm{diag}(r_1, r_2, \cdots, r_v) - \frac{1}{k} LL' - \frac{1}{b} MM' + \frac{1}{bk} \underset{\sim}{r}\underset{\sim}{r}' , \tag{1.1}$$

where $\underset{\sim}{r}' = (r_1, r_2, \cdots, r_v)$.

The design is said to be row-treatment connected if $R(C_1) = v-1$, column-treatment connected if $R(C_2) = v-1$, column-row connected if $R(C_3) = k-1$, and doubly connected if $R(C^*) = v-1$. It may be noted that a doubly connected design need not necessarily ensure the estimation of all elementary contrasts of row and column effects.

This paper is a contribution toward the theory of the doubly-connectedness property of two-way elimination of heterogeneity designs.

2. A necessary and sufficient condition for doubly-connectedness of ordinary two-way elimination of heterogeneity designs. Before we state and prove our main theorem, we state the following lemma whose proof is obvious.

Lemma 2.1. The C-matrices, C_1, C_2, C^* have rank v-1 if and only if $C_1 + aJ_{v,v}$, $C_2 + aJ_{v,v}$, $C^* + aJ_{v,v}$ are non-singular, where a is a non-zero scalar and $J_{m,n}$ is an m X n matrix with 1 everywhere.

Theorem 2.1. *In an ordinary two-way elimination of heterogeneity design, let $r_1 = r_2 = \cdots = r_v = r$ and let $L'M = rJ_{b,k}$. Then the design is doubly-connected if and only if it is row-treatment and column-treatment connected.*

Proof. Under the assumptions of the theorem

$$C^* = rC_1C_2 , \tag{2.1}$$

$$C^* + aJ_{v,v} = r\left(C_1 + \sqrt{\frac{a}{rv}}\, J_{v,v}\right)\left(C_2 + \sqrt{\frac{a}{rv}}\, J_{v,v}\right) \tag{2.2}$$

for any non-zero real a . Thus the following two-way relations establishing the theorem hold:

The design is doubly-connected $\Leftrightarrow R\left(C^*\right) = v-1 \Leftrightarrow C^* + aJ_{v,v}$ is non-singular

$$\Leftrightarrow C_1 + \sqrt{\frac{a}{rv}}J_{v,v} \text{ and } C_2 + \sqrt{\frac{a}{rv}}J_{v,v} \text{ are non-singular}$$

$$\Leftrightarrow R\left(C_1\right) = v-1 \text{ and } R\left(C_2\right) = v-1$$

$\Leftrightarrow$ The design is row-treatment and column-treatment connected.

The conditions of the theorem are certainly much stronger than what is needed, for the design

$$(2.3) \qquad \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{array}$$

is row-treatment, column-treatment, row-column and doubly-connected without satisfying the assumptions of the theorem. However, some conditions are needed as the following design

$$(2.4) \qquad \begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array}$$

does not satisfy the assumptions or the conclusions of the theorem.

3. An algorithm for constructing series of doubly-disconnected designs, which are pairwise connected. Consider an $s \times s$ latin square design where the diagonal elements are $1,2,\cdots,s$ in that order. When $s \neq 2$, the construction of such designs, in general, was provided by Hedayat and Federer [1970]. By omitting the diagonal treatments excepting the first one, we obtain a doubly-disconnected design, which is pairwise connected with respect to row, column, and treatments. In fact if I_n is an identity matrix, for such a design

$$L = M = N = L' = M' = N' = \begin{bmatrix} 1 & J_{1,s-1} \\ J_{s-1,1} & J_{s-1,s-1} - I_{s-1} \end{bmatrix}$$

$$u_1 = w_1 = r_1 = s$$

$$u_2 = \cdots = u_s = w_2 = \cdots = w_s = r_2 = \cdots = r_s = s-1$$

$$C_1 = C_2 = C_3 = M\text{-}L \operatorname{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_s}\right) N$$

so that

$$C^* = C_1 - C_3C_3^-C_3 = C_1 - C_3 = 0_{s,s},$$

where $0_{s,s}$ is the null matrix of order s .

4. A necessary condition for doubly-connectedness in generalized two-way elimination of heterogeneity designs. Contrary to the belief that a doubly-connected design is pairwise connected, the following result holds:

Theorem 4.1. *A doubly-connected generalized two-way elimination of heterogeneity design is row-treatment and column-treatment connected.*

Proof. Let if possible a doubly-connected design be row-treatment disconnected. Then there exists a $v \times 1$ column vector ξ orthogonal to $J_{v,1}$ such that $C_1\xi = 0_{v,1}$. Then, we have

$$\text{(4.1)} \qquad \xi' C^* \xi = - \left\{\left(M\text{-}L \operatorname{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)\xi\right\}' C_3^- \times \left\{\left(M\text{-}L \operatorname{diag}\left(\frac{1}{u_1}, \frac{1}{u_2}, \cdots, \frac{1}{u_b}\right) N\right)\xi\right\} ,$$

which is a contradiction as the left-hand side is a positive quantity and the right-hand side is a non-positive quantity in view of C^* and C_3^- being positive semi-definite matrices. Thus a doubly-connected design is row-treatment connected. Analogously, one may show it to be column-treatment connected.

The above result can also be obtained from the model of such designs. A doubly-connected design need not necessarily be row-column connected. The following design where X's indicate blanks is doubly-connected and hence row-treatment and column-treatment connected, but is row-column disconnected:

1	2	X	X
2	1	X	X
X	X	1	2
X	X	2	1

In fact, the class of doubly-connected designs $I \underset{-}{\bar{X}} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is row-column disconnected, where I is the identity matrix and $\underset{-}{\bar{X}}$ is the symbol for the Kronecker product of matrices.

5. <u>A property of doubly-connected designs</u>. One may wonder whether a doubly-connected design can be used to estimate every elementary contrast of row and column effects. The answer is provided by the following theorem:

<u>Theorem 5.1. A doubly-connected design, which is also row-column connected, provides estimates for every elementary contrast of row and column effects.</u>

<u>Proof</u>. For a doubly-connected design which is row-column connected, the following hold:

$$\begin{aligned} R\left(C^*\right) &= R\left(C_1\right) = R\left(C_2\right) = v-1 \\ R\left(C_3\right) &= k-1 \qquad (5.1) \\ R\left(C_4\right) &= b-1 , \end{aligned}$$

and thus

$$R\begin{bmatrix} \operatorname{diag}(u_1,u_2,\cdots,u_b) & N & L' \\ N' & \operatorname{diag}(w_1,w_2,\cdots,w_k) & M' \\ L & M & \operatorname{diag}(r_1,r_2,\cdots,r_v) \end{bmatrix}$$

(5.2)

$$= R\begin{bmatrix} \operatorname{diag}(u_1,u_2,\cdots,u_b) & N & L' \\ 0 & C_3 & * \\ 0 & 0 & C^* \end{bmatrix}$$

$$= b + k - 1 + v - 1 = v + b + k - 2 ,$$

where * denotes a matrix obtained in the sweeping-out process. Again

$$v+b+k-2 = R\begin{bmatrix} \operatorname{diag}(u_1,u_2,\cdots,u_b) & N & L' \\ N' & \operatorname{diag}(w_1,w_2,\cdots,w_k) & M' \\ L & M & \operatorname{diag}(r_1,r_2,\cdots,r_v) \end{bmatrix}$$

(5.3)

$$= R\begin{bmatrix} C_4^* & 0 & 0 \\ N' & \operatorname{diag}(w_1,w_2,\cdots,w_k) & M' \\ * & 0 & C_2 \end{bmatrix}$$

$$= R\left(C_4^*\right) + k + v - 1 ,$$

from which it follows that

(5.4) $$R\left(C_4^*\right) = b-1 ,$$

and

$$v+b+k-2 = R\begin{bmatrix} \mathrm{diag}\left(u_1,u_2,\cdots,u_b\right) & N & L' \\ N' & \mathrm{diag}\left(w_1,w_2,\cdots,w_k\right) & M' \\ L & M & \mathrm{diag}\left(r_1,r_2,\cdots,r_v\right) \end{bmatrix}$$

(5.5)

$$= R\begin{bmatrix} \mathrm{diag}\left(u_1,u_2,\cdots,u_b\right) & N & L' \\ 0 & C_3^* & 0 \\ 0 & * & C_1 \end{bmatrix}$$

$$= R\left(C_3^*\right) + b + v - 1 \; ,$$

from which it follows that

$$R\left(C_3^*\right) = k-1 \quad . \tag{5.6}$$

Thus all elementary contrasts of row and column effects are estimable establishing the theorem.

6. Concluding remarks: Though the results in this paper are obtained for generalized two-way elimination of heterogeneity designs, they can be translated into the terminology of any 3-factor experiment without any loss of generality in an obvious way.

7. Acknowledgement. The authors wish to thank A. Hedayat for many helpful discussions.

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